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FIGURATE SERIES.

By B. B. Smyth, Topeka.

Between arithmetical and geometrical series are to be found many series known as figurate series, based upon the fact that they represent geometrical figures, such as triangles, squares, pentagons, hexagons, octagons, tetrahedrons, hexahedrons, cubes, parallelopipeds, pyramids, cones, cylinders, spheres, and frustums of pyramids and cones.

Figurate series are produced by addition, to each of the successive terms, of a constantly increasing number, usually the preceding term, the last preceding term but one, the same term of another series, or some other definite term, as the case may require.

Figurate series are of two kinds—planimetric and volumetric, the former secondary, the latter tertiary. Arithmetical series are primary and linear.

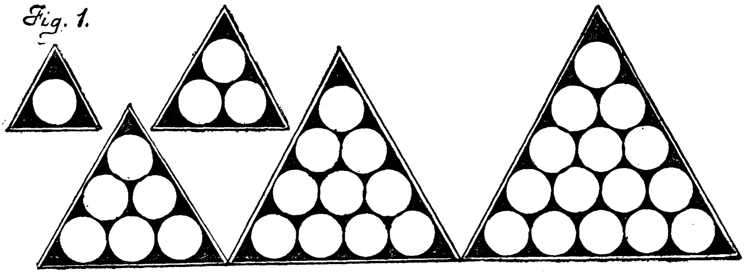
In most of the following series each term is found by adding to the same term of the preceding series the preceding term of the same series. It is equivalent to the sum of all the terms of the preceding series up to and including the same term. Thus the seventh term (84) of the tetrahedral series is equal to the sum of the first seven terms of the trigonal series; the eighth term (64) of the square series is equal to the first eight terms of the alternate series. In the following examples linear series are marked (1), planimetric series are marked (2), and volumetric series are marked (3).

TRIGONAL SERIES.

Terms:	1st.	2d.	3d.	4th.	5th.	6th.	7th.	8th.	9th.	10th, etc.	
		1	1	1	1	1	1	1	1	1	
(1)	1	2	3	4	5	6	7	8	9	10	Numeral Series.
(2)	1	3	6	10	15	21	28	36	45	55	Trigonal Series.
(3)	1	4	10	20	35	56	84	120	165	220	Tetrahedral Series.
Add		1	4	10	20	35	56	84	120	165	" "
(3)	1	5	14	30	55	91	140	204	285	385	Hexahedral Series.

This tetrahedron is, of course, a regular equilateral triangular pyramid. The hexahedron is not a cube but a double triangular pyramid. Any term in the hexahedral series is obtained by addition of two terms of the tetrahedral series—a similar term with the preceding term.

Fig. 1.



SQUARE SERIES.

Terms:	1st.	2d.	3d.	4th.	5th.	6th.	7th.	8th.	9th.	10th, etc.	
		2	2	2	2	2	2	2	2	2	
(1)	1	3	5	7	9	11	13	15	17	19	.... Alternate Series.
(2)	1	4	9	16	25	36	49	64	81	100	.... Square Series.
(3)	1	5	14	30	55	91	140	204	285	385	.... Pyramidal Series.
		1	5	14	30	55	91	140	204	285	.... " "
(3)	1	6	19	44	85	146	231	344	489	670	.... Octahedral Series.

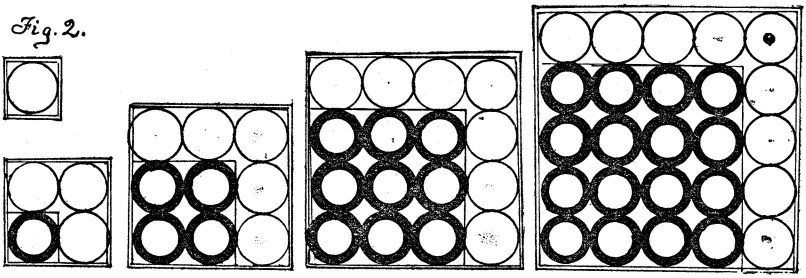
The units of the hexahedral and pyramidal series, though the same, are differently arranged. For example: In the fifth pyramid the lower tier contains 5 × 5 = 25 units. The second tier, laid in the interstices on top of this, contains 4 × 4 = 16 units; the third tier, 9; the fourth, 4; and on top, 1; making in all, 55 units.

Now compare the fifth hexahedron. At the bottom there is 1 unit. Just above it there are 3 in a triangle. The next layer or tier has 6 in a triangle; the fourth tier has 10; and the fifth, 15. Then they decrease to the top. The sixth tier has 10, as has the fourth; the seventh has 6, like the third; the eighth has 3, the same as the second; and on top is 1, as at the bottom. Thus there are 55 in all, the same as in the pyramid.

These forms, for school work, may be cut out of cardboard, potatoes, etc., or molded from wax, clay, or other plastic substance, or built up of marbles or wooden balls. A very good series may be made in outline by using wooden toothpicks, fastening together at the tips with small cubes of potato. In a few days the bits of potato dry hard and hold the sticks firmly in position.

The method of obtaining the square series from the alternate series, as above, may be represented to the eye thus, in which the light circles show the successive additions:

Fig. 2.



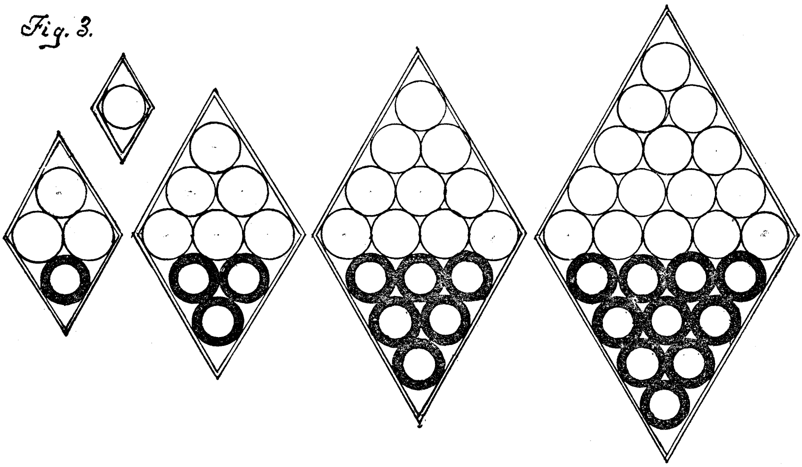
There are several other methods of obtaining the square series by addition, thus:

Secondly. By adding to each term of the trigonal series the preceding term, as:

	0	1	3	6	10	15	21	28	36	45	55	66 ....
Add		0	1	3	6	10	15	21	28	36	45	55 ....
	0	1	4	9	16	25	36	49	64	81	100	121 ....

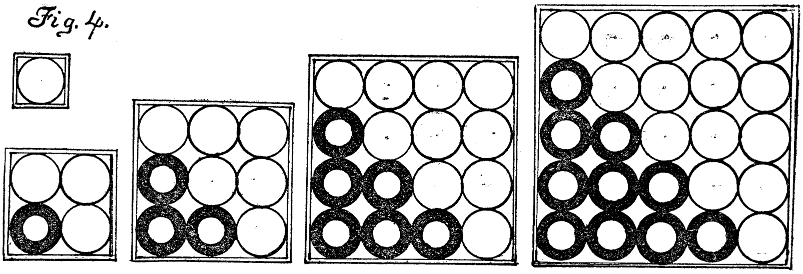
This would appear to the eye as rhombs, thus:

Fig. 3.



These, by a simple readjustment of the units, become squares, thus:

Fig. 4.



Thirdly. By subtracting from the tetrahedral series the second preceding term, thus:

	0	1	4	10	20	35	56	84	120	165	220	286	364 . . .
Subtract			0	1	4	10	20	35	56	84	120	165	220 . .
	0	1	4	9	16	25	36	49	64	81	100	121	144 . . .

Fourthly. By subtracting from the hexahedral series the preceding term, thus:

	0	1	5	14	30	55	91	140	204	285	385	506	650 . . . .
Subtract		0	1	5	14	30	55	91	140	204	285	385	506 . . . .
	0	1	4	9	16	25	36	49	64	81	100	121	144 . . . .

Remembering that the units in the pyramidal series are the same as in the hexahedral series, if we consider that the above series is a pyramidal series, the apparent strangeness of the performance disappears.

The square series seems to be indestructible by addition to it (or subtraction from it) of alternate, quartate, sextate, octate, or any regular arithmetical series, in which the common difference is 2 or any multiple of 2.

CASE I. EVEN SERIES.

Common difference of series, 4.

....	16	9	4	1	0	1	4	9	16	25	36	49	64	81	100	....
....	-12	-8	-4	0	4	8	12	16	20	24	28	32	36	40	44	....
....	4	1	0	1	4	9	16	25	36	49	64	81	100	121	144	....

Common difference, 8.

....	36	25	16	9	4	1	0	1	4	9	16	25	36	49	64	....
....	-32	-24	-16	-8	0	8	16	24	32	40	48	56	64	72	80	....
....	4	1	0	1	4	9	16	25	36	49	64	81	100	121	144	....

Common difference, 12.

....	64	49	36	25	16	9	4	1	0	1	4	9	16	25	36	....
....	-60	-48	-36	-24	-12	0	12	24	36	48	60	72	84	96	108	....
....	4	1	0	1	4	9	16	25	36	49	64	81	100	121	144	....

Common difference, 20.

....	144	121	100	81	64	49	36	25	16	9	4	1	0	1	4	....
....	-140	-120	-100	-80	-60	-40	-20	0	20	40	60	80	100	120	140	....
....	4	1	0	1	4	9	16	25	36	49	64	81	100	121	144	....

Common difference, 32.

....	100	81	64	49	36	25	16	9	4	1	0	1	4	....
....	-64	-32	0	32	64	96	128	160	192	224	256	288	320	....
....	36	49	64	81	100	121	144	169	196	225	256	289	324	....

Common difference, 60.

....	400	361	324	289	256	225	196	169	144	121	100	81	....
....	-300	-240	-180	-120	-60	0	60	120	180	240	300	360	....
....	100	121	144	169	196	225	256	289	324	361	400	441	....

Common difference, 100.

....	900	841	784	729	676	625	576	529	484	441	400	361	....
....	-500	-400	-300	-200	-100	0	100	200	300	400	500	600	....
....	400	441	484	529	576	625	676	729	784	841	900	961	....

CASE II. ODD SERIES.

Common difference, 2.

....	9	4	1	0	1	4	9	16	25	36	49	64	81	100	121	144	....
....	-5	-3	-1	1	3	5	7	9	11	13	15	17	19	21	23	25	....
....	4	1	0	1	4	9	16	25	36	49	64	81	100	121	144	169	....

Common difference, 6.

....	9	4	1	0	1	4	9	16	25	36	49	64	81	100	121	....
....	-9	-3	3	9	15	21	27	33	39	45	51	57	63	69	75	....
....	0	1	4	9	16	25	36	49	64	81	100	121	144	169	196	....

Common difference, 10.

....	25	16	9	4	1	0	1	4	9	16	25	36	49	64	81	....
....	-25	-15	-5	5	15	25	35	45	55	65	75	85	95	105	115	....
....	0	1	4	9	16	25	36	49	64	81	100	121	144	169	196	....

Common difference, 14.

....	49	36	25	16	9	4	1	0	1	4	9	16	25	36	49	....
....	-49	-35	-21	-7	7	21	35	49	63	77	91	105	119	133	147	....
....	0	1	4	9	16	25	36	49	64	81	100	121	144	169	196	....

From these examples we may deduce the following formula:

Let *cd* be the common difference of the series to be added; then, having arranged the square series as far as desired in either direction from *0*, $\left[\frac{cd}{2}\right]^2$ is to be added to *0*, and $-\left[\frac{cd}{2}\right]^2$ to be added to the term in the square series equal to $\left[\frac{cd}{2}\right]^2$, so as to make the sum of the two *0*. The other terms of the series to be added will be $\left[\frac{cd}{2}\right]^2 \pm \underline{cd}$, $\left[\frac{cd}{2}\right]^2 \pm \underline{2cd}$, $\left[\frac{cd}{2}\right]^2 \pm \underline{3\,cd}$, etc.

In the even series *0* of the series to be added falls under $\left[\frac{cd}{4}\right]^2$ of the square series. In the odd series *0* in the series to be added does not occur; because, of course, *0* is not an odd number.

To show the harmonies that exist between the arithmetical series and the square series we will inspect a few of them. Take, for instance, the first example given, having a common difference of 4:

Squares	36	25	16	9	4	1	0	1	4	9	16	25	36	49
Quartate series	-20	-16	-12	-8	-4	0	4	8	12	16	20	24	28	32
Sums	16	9	4	1	0	1	4	9	16	25	36	49	64	81
Remainders	56	41	28	17	8	1	-4	-7	-8	-7	-4	1	8	17
Difference	-40	-32	-24	-16	-8	0	8	16	24	32	40	48	56	64
Common difference	8	8	8	8	8	8	8	8	8	8	8	8	8	
Difference of rem.	15	13	11	9	7	5	3	1	-1	-3	-5	-7	-9	
Difference of sums	7	5	3	1	-1	-3	-5	-7	-9	-11	-13	-15	-17	
Differences	8	8	8	8	8	8	8	8	8	8	8	8	8	

Take another example, with a common difference of 2:

Squares	25	16	9	4	1	0	1	4	9	16	25	36	49	64
Alternate series,	-9	-7	-5	-3	-1	1	3	5	7	9	11	13	15	17
Sums	16	9	4	1	0	1	4	9	16	25	36	49	64	81
Remainders	34	23	14	7	2	-1	-2	-1	2	7	14	23	34	47
Difference	-18	-14	-10	-6	-2	2	6	10	14	18	22	26	30	34
Common diff.	4	4	4	4	4	4	4	4	4	4	4	4	4	
Diff. of rem.	11	9	7	5	3	1	-1	-3	-5	-7	-9	-11	-13	
Diff. of sums	7	5	3	1	-1	-3	-5	-7	-9	-11	-13	-15	-17	
Differences	4	4	4	4	4	4	4	4	4	4	4	4	4	

PENTAGONAL SERIES.

		5	5	5	5	5	5	5	Constant.
	0	5	10	15	20	25	30	35	Pentate series.
	1	6	16	31	51	76	106	141	Pentagonal series.
Add	0	1	2	3	4	5	6	7	
	1	7	18	34	55	81	112	148	
× 40 =	40	280	720	1360	2200	3240	4480	5920	
+ 9 =	49	289	729	1369	2209	3249	4489	5929	Square series.
$\sqrt{}$ =	7	17	27	37	47	57	67	77	

Thus even the pentagonal series, under certain manipulation, is resolved into the squares of a certain series regulated by the constant 5.

HEXAGONAL SERIES.

The method of obtaining this series is shown under "cubic series."

CUBIC SERIES.

Terms:	1st.	2d.	3d.	4th.	5th.	6th.	7th.	8th.	9th.	10th, etc.	
		6	6	6	6	6	6	6	6	6.....	Constant.
(1)	0	6	12	18	24	30	36	42	48	54.....	Sextate Series.
(2)	1	7	19	37	61	91	127	169	217	271.....	Hexagonal Series.
(3)	1	8	27	64	125	216	343	512	729	1000.....	Cubic Series.
(3)	1	9	36	100	225	441	784	1296	2025	3025.....	Tower Series.

The cubic series is regular hexahedral, and differs from the hexahedral series before given, as explained under the head of "trigonal series."

The "tower" series represents piles of cubes of successively decreasing sizes. The tower series is equal to the square of the trigonal series.

To show how much the formation of squares and cubes depends on the trigonal and tetrahedral series, we will analyze a few of each:

SQUARES.

$1 = (0 \times 8) + 1 = 1^2$
$1 \times 8 = \underline{8} \quad \underline{1}$
$9 = (1 \times 8) + 1 = 3^2$
$2 \times 8 = \underline{16} \quad \underline{2}$
$25 = (3 \times 8) + 1 = 5^2$
$3 \times 8 = \underline{24} \quad \underline{3}$
$49 = (6 \times 8) + 1 = 7^2$
$4 \times 8 = \underline{32} \quad \underline{4}$
$81 = (10 \times 8) + 1 = 9^2$
$5 \times 8 = \underline{40} \quad \underline{5}$
$121 = (15 \times 8) + 1 = 11^2$
$6 \times 8 = \underline{48} \quad \underline{6}$
$169 = (21 \times 8) + 1 = 13^2$
$7 \times 8 = \underline{56} \quad \underline{7}$
$225 = (28 \times 8) + 1 = 15^2$
$8 \times 8 = \underline{64} \quad \underline{8}$
$289 = (36 \times 8) + 1 = 17^2$
$9 \times 8 = \underline{72} \quad \underline{9}$
$361 = (45 \times 8) + 1 = 19^2$
$10 \times 8 = \underline{80} \quad \underline{10}$
$441 = (55 \times 8) + 1 = 21^2$
$11 \times 8 = \underline{88} \quad \underline{11}$
$529 = (66 \times 8) + 1 = 23^2$
$12 \times 8 = \underline{96} \quad \underline{12}$
$625 = (78 \times 8) + 1 = 25^2$
$13 \times 8 = \underline{104} \quad \underline{13}$
$729 = (91 \times 8) + 1 = 27^2$
$14 \times 8 = \underline{112} \quad \underline{14}$
$841 = (105 \times 8) + 1 = 29^2$
$15 \times 8 = \underline{120} \quad \underline{15}$
$961 = (120 \times 8) + 1 = 31^2$
$16 \times 8 = 128$

$0^2 = (0 \times 8) + (0 \times 4) = 0$
$\underline{0} \quad \underline{4} = (0 \times 8) + 4$
$2^2 = (0 \times 8) + (1 \times 4) = 4$
$\underline{1} \quad \underline{12} = (1 \times 8) + 4$
$4^2 = (1 \times 8) + (2 \times 4) = 16$
$\underline{2} \quad \underline{20} = (2 \times 8) + 4$
$6^2 = (3 \times 8) + (3 \times 4) = 36$
$\underline{3} \quad \underline{28} = (3 \times 8) + 4$
$8^2 = (6 \times 8) + (4 \times 4) = 64$
$\underline{4} \quad \underline{36} = (4 \times 8) + 4$
$10^2 = (10 \times 8) + (5 \times 4) = 100$
$\underline{5} \quad \underline{44} = (5 \times 8) + 4$
$12^2 = (15 \times 8) + (6 \times 4) = 144$
$\underline{6} \quad \underline{52} = (6 \times 8) + 4$
$14^2 = (21 \times 8) + (7 \times 4) = 196$
$\underline{7} \quad \underline{60} = (7 \times 8) + 4$
$16^2 = (28 \times 8) + (8 \times 4) = 256$
$\underline{8} \quad \underline{68} = (8 \times 8) + 4$
$18^2 = (36 \times 8) + (9 \times 4) = 324$
$\underline{9} \quad \underline{76} = (9 \times 8) + 4$
$20^2 = (45 \times 8) + (10 \times 4) = 400$
$\underline{10} \quad \underline{84} = (10 \times 8) + 4$
$22^2 = (55 \times 8) + (11 \times 4) = 484$
$\underline{11} \quad \underline{92} = (11 \times 8) + 4$
$24^2 = (66 \times 8) + (12 \times 4) = 576$
$\underline{12} \quad \underline{100} = (12 \times 8) + 4$
$26^2 = (78 \times 8) + (13 \times 4) = 676$
$\underline{13} \quad \underline{108} = (13 \times 8) + 4$
$28^2 = (91 \times 8) + (14 \times 4) = 784$
$\underline{14} \quad \underline{116} = (14 \times 8) + 4$
$30^2 = (105 \times 8) + (15 \times 4) = 900$
$\underline{15} \quad \underline{124} = (15 \times 8) + 4$
$32^2 = (120 \times 8) + (16 \times 4) = 1024$

Here we see the constant 8 multiplied by the successive terms of the trigonal series, plus a certain number which is either constant (1 in the odd squares) or bears a constant relation to the root.

CUBES.

$0^3 = (\ 0 \times 6) + 0 =$	0	
(Add) $\underline{0}$	(Add) $\underline{1}$	$= (\ 0 \times 6) + 1$
$1^3 = (\ 0 \times 6) + 1 =$	1	(Add) $\underline{1}$
(Add) $\underline{1}$	(Add) $\underline{7}$	$= (\ 1 \times 6) + 1$
$2^3 = (\ 1 \times 6) + 2 =$	8	(Add) $\underline{2}$
$\underline{3}$	(Add) $\underline{19}$	$= (\ 3 \times 6) + 1$
$3^3 = (\ 4 \times 6) + 3 =$	27	(Add) $\underline{3}$
$\underline{6}$	$\underline{37}$	$= (\ 6 \times 6) + 1$
$4^3 = (\ 10 \times 6) + 4 =$	64	$\underline{4}$
$\underline{10}$	$\underline{61}$	$= (\ 10 \times 6) + 1$
$5^3 = (\ 20 \times 6) + 5 =$	125	$\underline{5}$
$\underline{15}$	$\underline{91}$	$= (\ 15 \times 6) + 1$
$6^3 = (\ 35 \times 6) + 6 =$	216	$\underline{6}$
$\underline{21}$	$\underline{127}$	$= (\ 21 \times 6) + 1$
$7^3 = (\ 56 \times 6) + 7 =$	343	$\underline{7}$
$\underline{28}$	$\underline{169}$	$= (\ 28 \times 6) + 1$
$8^3 = (\ 84 \times 6) + 8 =$	512	$\underline{8}$
$\underline{36}$	$\underline{217}$	$= (\ 36 \times 6) + 1$
$9^3 = (120 \times 6) + 9 =$	729	$\underline{9}$
$\underline{45}$	$\underline{271}$	$= (\ 45 \times 6) + 1$
$10^3 = (165 \times 6) + 10 =$	1000	$\underline{10}$
$\underline{55}$	$\underline{331}$	$= (\ 55 \times 6) + 1$
$11^3 = (220 \times 6) + 11 =$	1331	$\underline{11}$
$\underline{66}$	$\underline{397}$	$= (\ 66 \times 6) + 1$
$12^3 = (286 \times 6) + 12 =$	1728	

Thus we find that the cubes are equal to the successive terms of the tetrahedral series multiplied by the constant factor 6, plus the root. The differences between the cubes are equal to the successive terms of the trigonal series multiplied by 6, plus the constant 1. The differences themselves are the successive terms of the hexagonal series.

I cannot do better in closing this paper than to produce the beautiful

PHYLLOTACTIC SERIES.

										0	1
								0	1	1	2
						0	1	1	2	3	5
				0	1	1	2	3	5	8	13
		0	1	1	2	3	5	8	13	21	34
0	1	1	2	3	5	8	13	21	34	55	89
1	2	3	5	8	13	21	34	55	89	144	233

I have carried this to the twelfth term; and that is about as high and complex as our observations on phyllotaxy have gone. A more frequent method of arranging these numbers is to place them in the form of fractions, thus:

$$\frac{0}{1}\quad\frac{1}{2}\quad\frac{1}{3}\quad\frac{2}{5}\quad\frac{3}{8}\quad\frac{5}{13}\quad\frac{8}{21}\quad\frac{13}{34}\quad\frac{21}{55}\quad\frac{34}{89}\quad\frac{55}{144}\quad\frac{89}{233}$$

In which either term of any fraction is obtained by addition of the two preceding terms of the same denomination. This may not be considered exactly a figurate series; but in nature it expresses the most beautiful and symmetrical arrangement of leaves, scales, bracts, seeds, etc., on stem, branch, or fruit, as the leaves on a house-leek, the florets on a spike of petalostemon, the scales on the cup of an acorn or a cone of pine or spruce, the seeds of a magnolia cone or a strawberry, the florets and seeds on the head of a sunflower, etc. These numbers show two things—how leaves, etc., have the greatest possible distribution to sunlight and air, and how the greatest number of leaves, seeds, etc., may be packed into the smallest and most economical space, in bud and fruit. The fractions express the distance around a stem, cone, or capitulum, from one leaf or seed to the next above.